

HOMOGENEOUS HYPERSURFACES WITH ISOTROPY IN AFFINE FOUR-SPACE

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Abstract: We classify the non-degenerate homogeneous hypersurfaces in real and complex affine four-space whose symmetry group is at least four-dimensional.

1 Statement of results

	Equation	Basepoint	Parameter Restrictions	Dimension of Isotropy
#1	$W = XY + Z^2$	$(0, 0, 0, 0)$		4
#2	$W^2 = XY + Z^2 + 1$	$(1, 0, 0, 0)$		3
#3	$W = XY + Z^2 + X^3$	$(0, 0, 0, 0)$		2
#4	$W = XY + Z^2 + X^2Z + \alpha X^4$	$(0, 0, 0, 0)$	α arbitrary	1
#5	$W = XY + Z^2 + XZ^2$	$(0, 0, 0, 0)$		1
#6	$W^2 = XY + X^2Y + X^2Z$	$(1, 1, 0, 1)$		1
#7	$W = XY + Z^\alpha$	$(1, 0, 0, 1)$	$\alpha \neq 0, 1, 2$	1
#8	$W = XY + e^Z$	$(1, 0, 0, 0)$		1
#9	$W = XY + \log Z$	$(0, 0, 0, 1)$		1
#10	$W = XY + Z \log Z$	$(0, 0, 0, 1)$		1
#11	$W^2 = XY + Z^\alpha$	$(1, 0, 0, 1)$	$\alpha \neq 0, 1, 2$	1
#12	$W^2 = XY + e^Z$	$(1, 0, 0, 0)$		1
#13	$WZ = XY + Z^\alpha$	$(1, 0, 0, 1)$	$\alpha \neq 0, 1, 2$	1
#14	$WZ = XY + Z \log Z$	$(0, 0, 0, 1)$		1
#15	$WZ = XY + Z^2 \log Z$	$(0, 0, 0, 1)$		1
#16	$W = XY + Z^2 + X^\alpha$	$(1, 1, 0, 0)$	$\alpha \neq 0, 1, 2, 3$	1
#17	$W = XY + Z^2 + e^X$	$(1, 0, 0, 0)$		1
#18	$W = XY + Z^2 + \log X$	$(0, 1, 0, 0)$		1
#19	$W = XY + Z^2 + X \log X$	$(0, 1, 0, 0)$		1
#20	$W = XY + Z^2 + X^2 \log X$	$(0, 1, 0, 0)$		1

Each of the equations in this table defines, near its basepoint, a non-degenerate homogeneous hypersurface in complex affine four-space. This means that it

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may be analytically continued to an orbit of a Lie subgroup of the group of affine symmetries. In each case, the full symmetry group (the maximal such Lie subgroup) has dimension at least four. In other words, there is a non-trivial (positive dimension) Lie subgroup preserving the basepoint. Indeed, the dimension of this *isotropy* is as listed. Furthermore, this is a complete list:

Theorem 1 *Every homogeneous hypersurface with isotropy in complex affine four-space may, for a suitable choice of affine coördinate system, be found in the table above. Different entries in this table and different values of the parameter α define affinely distinct hypersurfaces.*

The corresponding real list is given at the end of this article. The complex classification list for non-degenerate surfaces in affine three-space has just three entries, namely the graph of a non-degenerate quadratic, the complex sphere, and the Cayley surface. This is proved explicitly in [1]. Theorem 1 will be deduced from an alternative formulation as follows.

Theorem 2 *Every homogeneous hypersurface with isotropy in complex affine four-space may be found in the following list of normal forms for a suitable choice of affine coördinates and free parameter b :*

$$\begin{aligned}
\mathbf{Qd} \quad w &= 2xy + z^2 + \mathcal{O}(5) = 2xy + z^2 \\
\mathbf{Sp} \quad w &= 2xy + z^2 + 4x^2y^2 + 4xyz^2 + z^4 + \mathcal{O}(5) = (1 - \sqrt{1 - 4(2xy + z^2)})/2 \\
\mathbf{I3} \quad w &= 2xy + z^2 + x^3 + \mathcal{O}(5) = 2xy + z^2 + x^3 \\
\mathbf{I2} \quad w &= 2xy + z^2 + x^2z + bx^4 + \mathcal{O}(5) = 2xy + z^2 + x^2z + bx^4 \\
\mathbf{I1.1} \quad w &= 2xy + z^2 + x^2y - 2xz^2 + \frac{1}{2}x^3y - x^2z^2 + \mathcal{O}(5) = \frac{4xy + 2z^2 - 5xz^2}{2 - x} \\
\mathbf{I1.2} \quad w &= 2xy + z^2 + x^2y - 2xz^2 + \frac{1}{2}x^3y + \frac{21}{4}x^2z^2 + \mathcal{O}(5) = \frac{4(2xy + z^2 + 5x^2y)}{(2 - x)(2 + 5x)} \\
\mathbf{I0.1} \quad w &= 2xy + z^2 + 3xyz - z^3 + \frac{9}{4}x^2y^2 - \frac{9}{2}xyz^2 + \frac{15}{32}bz^4 + \mathcal{O}(5) \\
\mathbf{I0.2} \quad w &= 2xy + z^2 + 3xyz - z^3 - \frac{9}{8}(5b - 7)x^2y^2 + \frac{9}{8}(5b - 9)xyz^2 + \frac{15}{32}bz^4 + \mathcal{O}(5) \\
\mathbf{I0.3} \quad w &= 2xy + z^2 + 3xyz - z^3 \\
&\quad - \frac{1}{4}(b + 1)(b + 7)x^2y^2 - \frac{1}{4}(b^2 - 7b - 26)xyz^2 - \frac{1}{16}(b^2 - 7b - 14)z^4 + \mathcal{O}(5) \\
\mathbf{Inr} \quad w &= 2xy + z^2 + x^3 + x^4 + bx^5 + \mathcal{O}(6)
\end{aligned}$$

In each case, the higher order terms are determined by the specified terms and no lower order truncation has this property. These hypersurfaces are affinely distinct save for the following three overlaps:

- Case **I0.1** and case **I0.2** agree when $b = 1$
- Case **I0.1** and case **I0.3** agree when $b = -4$
- Case **I0.2** and case **I0.3** agree when $b = 7/2$.

This list is obtained by choosing coördinates so that the Taylor series of the defining function and, consequently, the isotropy of the surface is in a preferred form. In principle, this gives an algorithm for locating the surface and its parameter as a local invariant. In cases **I0.1**, **I0.2**, and **I0.3** there is clearly some choice for what to take as parameter. For example, the coefficient of z^4 might be a more natural choice in case **I0.1**. The particular choices made are so that b is continuous across the three overlaps between these cases whilst at the same time not making any of the formulae too unwieldy.

The following table compares Theorems 1 and 2.

Explicit Form	Normal Form	How the Parameters α and b are Related
#1	Qd	
#2	Sp	
#3	I3	
#4	I2	$\alpha = b$
#5	I1.1	
#6	I1.2	
#7	I0.1	$\alpha = (2b - 2)/(b + 4)$
#8	I0.1 or I0.3	$b = -4$
#9	I0.1 or I0.2	$b = 1$
#10	I0.1	$b = 6$
#11	I0.2	$\alpha = (4b - 4)/(4b - 9)$
#12	I0.2	$b = 9/4$
#13	I0.3	$\alpha = 15/(b + 4)$
#14	I0.3	$b = 11$
#15	I0.2 or I0.3	$b = 7/2$
#16	Inr	$\alpha = (15b - 16)/(5b - 4)$
#17	Inr	$b = 4/5$
#18	Inr	$b = 16/15$
#19	Inr	$b = 6/5$
#20	Inr	$b = 8/5$

This article is organised as follows. In the next section we shall describe how to normalise up to third order the defining function of a non-degenerate hypersurface under the assumption that it is homogeneous with isotropy. Then, in §3 we shall use these normalisations to effect the classification. The method follows [2] and especially the criteria for homogeneity developed therein. The conversion of this classification to the list of explicit defining functions is described in §4. In

most of this article we are working for simplicity over the complex numbers. A real classification may be performed similarly. In §5 we make a few remarks on this task and list the real defining functions. Though the details are different, in [4] Loboda uses affine normal forms to classify homogeneous surfaces in affine three-space. Presumably, his approach could also be employed for hypersurfaces with isotropy. After our article was completed we learned of a manuscript by N. Mozhey who considers the same problem with a different method.

2 Normal forms

We shall choose affine coördinates so that the hypersurface Σ and its isotropy are in some preferred normal form. Firstly, we shall choose coördinates (x, y, z, w) so that Σ passes through the origin and so that $\{w = 0\}$ is its tangent plane. Recall that Σ is supposed non-degenerate. This allows us to normalise the quadratic terms of its defining function. Also, we shall take the w -axis to be the affine normal. The effect of these choices is that Σ may be defined by a power series

$$w = F(x, y, z) = 2xy + z^2 + \text{cubic terms} + O(4) \quad (1)$$

whose cubic terms are trace-free with respect to the quadratic form associated with $2xy + z^2$ (as explained in [3] or [2, Proposition 1]). Specifically, this means that the cubic terms are spanned by

$$x^3, x^2z, x^2y - 2xz^2, 3xyz - z^3, xy^2 - 2yz^2, y^2z, y^3. \quad (2)$$

At this stage, the remaining coördinate freedom is $O(3, \mathbb{C})$ acting on (x, y, z) together with the rescaling

$$x \mapsto \lambda x \quad y \mapsto \lambda y \quad z \mapsto \lambda z \quad w \mapsto \lambda^2 w. \quad (3)$$

The corresponding Lie algebra may be represented by matrices of the form

$$\begin{bmatrix} t-r & 0 & p & 0 \\ 0 & t+r & -q & 0 \\ q & -p & t & 0 \\ 0 & 0 & 0 & 2t \end{bmatrix} \quad (4)$$

with the usual Lie bracket of matrices. As far as $O(3, \mathbb{C})$ is concerned, the adjoint action for $\mathfrak{o}(3, \mathbb{C})$

$$\left[\begin{bmatrix} -c & 0 & a \\ 0 & c & -b \\ b & -a & 0 \end{bmatrix}, \begin{bmatrix} -r & 0 & p \\ 0 & r & -q \\ q & -p & 0 \end{bmatrix} \right] = \begin{bmatrix} aq - bp & 0 & ar - cp \\ 0 & bp - aq & br - cq \\ cq - br & cp - ar & 0 \end{bmatrix}$$

may be viewed as matrix multiplication

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} \longmapsto \begin{bmatrix} -c & 0 & a \\ 0 & c & -b \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

and similarly for the Adjoint action

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} \mapsto M \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad \text{for } M \in O(3, \mathbb{C}).$$

This standard representation has, up to scale, three orbits, namely the origin, the vectors of non-zero length, and the non-zero null vectors. Accordingly, we may conjugate and rescale any matrix in $\mathfrak{o}(3, \mathbb{C})$ into one of three standard forms:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}. \quad (5)$$

We are now in a position to normalise the isotropy of Σ . Any 1-parameter subgroup of this isotropy will be generated by a matrix of the form (4) and the $\mathfrak{o}(3, \mathbb{C})$ component thereof may then be normalised as above. In the second two cases this fixes the scaling. Therefore, we obtain three possibilities

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} t-1 & 0 & 0 & 0 \\ 0 & t+1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 2t \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & -1 & 0 \\ 1 & 0 & t & 0 \\ 0 & 0 & 0 & 2t \end{bmatrix} \quad (6)$$

which we shall take as normal forms for an isotropy generator. The first possibility occurs but is very restrictive. It means that the whole power series (1) is preserved by the rescaling (3). This forces all cubic and higher terms to vanish. We are left with a quadratic defining function. This is case **Qd** of Theorem 2. Its isotropy is four-dimensional, generated by $O(3, \mathbb{C})$ and (3). One more case which can be dealt with separately is when all cubic terms vanish but there are some non-vanishing higher order terms. That it must be the complex hypersphere (case **Sp**) is an immediate consequence of the classical Maschke-Pick-Berwald Theorem (see, e.g. [6]) which states that a non-degenerate hypersurface with vanishing cubic form is a hyperquadric. This classical theorem does not assume *a priori* that the hypersurface is homogeneous. When homogeneity is assumed the conclusion is also an easy consequence of our approach (see §3).

We may now suppose that there are non-zero cubic terms in (1) and investigate the consequences of the corresponding hypersurface Σ admitting either of the second two of (6) as a symmetry. Writing $c(x, y, z)$ for the cubic terms, this means that

$$\left[(t-1)x \frac{\partial}{\partial x} + (t+1)y \frac{\partial}{\partial y} + tz \frac{\partial}{\partial z} \right] c(x, y, z) - 2tc(x, y, z) = 0$$

or

$$\left[tx \frac{\partial}{\partial x} + (ty - z) \frac{\partial}{\partial y} + (x + tz) \frac{\partial}{\partial z} \right] c(x, y, z) - 2tc(x, y, z) = 0,$$

respectively. Using (2) as a basis of the cubic terms, these equations place $c(x, y, z)$ in the kernel of the matrices

$$\begin{bmatrix} t-3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t-2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t+2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t+3 \end{bmatrix}$$

or

$$\begin{bmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & t \end{bmatrix},$$

respectively. The first matrix is singular if and only if t is an integer in the range $-3, \dots, 3$ whilst the second is singular only for $t = 0$. We conclude that normalising the isotropy as we have done automatically forces $c(x, y, z)$ to be a simple multiple of one of the seven basic cubics (2). Swopping x and y if necessary, we have almost proved the following.

Theorem 3 *A homogeneous hypersurface with isotropy in complex affine four-space may be locally defined for a suitable choice of affine coördinate system by a power series of the form (1) which, if the cubic terms are non-zero, may be further normalised to have one of the following forms:*

I3 $w = 2xy + z^2 + x^3 + O(4)$

I2 $w = 2xy + z^2 + x^2z + O(4)$

I1 $w = 2xy + z^2 + x^2y - 2xz^2 + O(4)$

I0 $w = 2xy + z^2 + 3xyz - z^3 + O(4).$

The residual coördinate freedom is generated by

$$x \mapsto \lambda^{t-1}x \quad y \mapsto \lambda^{t+1}y \quad z \mapsto \lambda^t z \quad w \mapsto \lambda^{2t}w \quad (7)$$

for $t = 3, 2, 1, 0$ respectively and, in addition,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ -t^2/2 & 1 & -t & 0 \\ t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

*in case **I3** and swopping x and y in case **I0**.*

Proof. It is easy to verify that these normal forms do, indeed, have the residual coördinate freedoms as stated and it remains to show that this is full extent thereof.

In case **I3** we can use the scaling freedom (7) to suppose that w is preserved on the nose, not merely up to scale. From the cubic term, x is then preserved up to scaling by a cube root of unity. This too may be incorporated into (7) and we may now suppose that x is also preserved on the nose. We are left with $O(3, \mathbb{C})$ transformations fixing a null vector and it is easy to check that they are of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ -t^2/2 & 1 & \mp t \\ t & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \quad (8)$$

With the positive sign this is a *null rotation* [5, p. 28]. The negative sign may be absorbed into a scaling (7) with $\lambda = -1$. Assembling these possibilities yields precisely the freedom as stated.

In all cases w and its axis are preserved up to scale and in case **I2** the cubic term ensures that x and z are also preserved up to scale:

$$x \mapsto \lambda x \quad z \mapsto \nu z \quad w \mapsto \kappa w \quad \text{where } \lambda^2 \nu = \kappa.$$

Now the quadratic terms force

$$y \mapsto \mu y \quad \text{where } \lambda \mu = \nu^2 = \kappa.$$

This is of the form (7).

Writing the cubic terms in case **I1** as $x(xy - 2z^2)$, a product of irreducibles, it is clear that x must be preserved up to scale as must the quadratic form $xy - 2z^2$. With the quadratic form $2xy + z^2$ also being preserved, this easily implies that y and z are now preserved up to scale. The result is of the form (7).

In case **I0**, the cubic terms factorise as $(3xy - z^2)z$ and similar reasoning implies that z and the quadratic form xy are preserved, firstly up to scale and then, by comparing scales, absolutely. The only remaining freedom is $O(2, \mathbb{C})$ acting in the (x, y) -variables. The identity connected component has the form (7) and the rest is generated by the reflection which swaps x and y . \square

3 Proof of Theorem 2

The proof is based on the criteria for homogeneity developed in [2]. For any formal power series or polynomial $G(x, y, z)$, we shall write $\text{Tr}^N G(x, y, z)$ for the polynomial obtained by truncation at order N :

$$\text{if } G(x, y) = \sum_{i,j,k=0}^{\infty} c_{ijk} x^i y^j z^k \quad \text{then} \quad \text{Tr}^N G(x, y) = \sum_{i+j+k \leq N} c_{ijk} x^i y^j z^k.$$

Theorem 4 Suppose $f(x, y, z)$ is a polynomial of degree N without constant or linear terms. If $f(x, y, z)$ can be completed to a formal power series whose graph near the origin is an open subset of a homogeneous hypersurface Σ , then there are 4×4 matrices P, Q, R such that

$$\begin{aligned} \text{Tr}^{N-1} \left[\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z), -1 \right] P \begin{bmatrix} x \\ y \\ z \\ f(x, y, z) \end{bmatrix} &= -\frac{\partial f}{\partial x}(x, y, z) \\ \text{Tr}^{N-1} \left[\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z), -1 \right] Q \begin{bmatrix} x \\ y \\ z \\ f(x, y, z) \end{bmatrix} &= -\frac{\partial f}{\partial y}(x, y, z) \\ \text{Tr}^{N-1} \left[\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z), -1 \right] R \begin{bmatrix} x \\ y \\ z \\ f(x, y, z) \end{bmatrix} &= -\frac{\partial f}{\partial z}(x, y, z). \end{aligned} \quad (9)$$

Conversely, suppose that these equations have solutions P, Q, R and that, for the general such solutions,

$$\text{Tr}^N \left[\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z), -1 \right] X \begin{bmatrix} x \\ y \\ z \\ f(x, y, z) \end{bmatrix} = 0 \quad (10)$$

for all X of the following three forms (where $P = (p_{i,j})$ etcetera):

$$\begin{aligned} X &= PQ - QP - (p_{1,2} - q_{1,1})P - (p_{2,2} - q_{2,1})Q - (p_{3,2} - q_{3,1})R \\ X &= QR - RQ - (q_{2,3} - r_{2,2})Q - (q_{3,3} - r_{3,2})R - (q_{1,3} - r_{1,2})P \\ X &= RP - PR - (r_{3,1} - p_{3,3})R - (r_{1,1} - p_{1,3})P - (r_{2,1} - p_{2,3})Q. \end{aligned}$$

Then $f(x, y, z)$ can be uniquely completed to a formal power series whose graph near the origin is an open subset of a homogeneous hypersurface. Furthermore, all homogeneous hypersurfaces in affine four-space arise in this way.

Proof. The proof is a simple modification of the corresponding result for surfaces proved in Theorem 1 and Corollary 1 of [2]. Suffice it to say that (9), for sufficiently large N , defines the symmetry algebra of Σ . That there are solutions is to say that there are infinitesimal symmetries in each of the three basic coördinate directions. This must be the case if Σ is homogeneous. For (10) to hold for X 's made out of the general P, Q, R is to say that this linear subspace of the Lie algebra of affine motions is closed under Lie bracket. Once the symmetry algebra has closed in this way, the higher order terms in the power series expansion of the defining function are completely pinned down (either by exponentiating to a Lie subgroup whose orbit is Σ or term-by-term from (9) now regarded as a series of equations for the coefficients of this power series with P, Q, R fixed). \square

The criteria in this theorem may be employed as follows. According to §2 and especially Theorem 3, the defining equation of a homogeneous hypersurface with isotropy may be normalised to third order. It is possible that all cubic terms vanish in which case we may consider the consequences of Theorem 4 for $f(x, y, z) = 2xy + z^2$ with $N = 3$. Otherwise, we can take $f(x, y, z)$ to be one of

$$\begin{array}{ll} \mathbf{I3} & 2xy + z^2 + x^3 \\ \mathbf{I2} & 2xy + z^2 + x^2z \\ \mathbf{I1} & 2xy + z^2 + x^2y - 2xz^2 \\ \mathbf{I0} & 2xy + z^2 + 3xyz - z^3. \end{array}$$

By way of illustration, let us consider in detail the case **I1** which is of medium difficulty. There are several computations carried out with the aid of MAPLE. Further details on this use of computer algebra will be given shortly.

The equations (9) are polynomial in x, y, z and so each coefficient must vanish separately. In addition, since we are searching for a hypersurface admitting

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad (11)$$

as a generator of isotropy, we may normalise P, Q, R by supposing that

$$p_{2,2} = 0 \quad q_{2,2} = 0 \quad r_{2,2} = 0. \quad (12)$$

Altogether, this gives a system of linear equations for the entries of P, Q, R which is easily solved:

$$\begin{aligned} P &= \begin{bmatrix} 2p_{3,3} - 3 & 0 & p_{1,3} & p_{1,4} \\ 0 & 0 & p_{2,3} & p_{2,4} \\ -p_{2,3} & -p_{1,3} & p_{3,3} & p_{3,4} \\ 0 & 2 & 0 & 2p_{3,3} - 2 \end{bmatrix} & Q &= \begin{bmatrix} 2q_{3,3} & 0 & q_{1,3} & q_{1,4} \\ -1/2 & 0 & q_{2,3} & q_{2,4} \\ -q_{2,3} & -q_{1,3} & q_{3,3} & q_{3,4} \\ 2 & 0 & 0 & 2q_{3,3} \end{bmatrix} \\ R &= \begin{bmatrix} 2r_{3,3} & 0 & r_{1,3} & r_{1,4} \\ 0 & 0 & r_{2,3} & r_{2,4} \\ 2 - r_{2,3} & -r_{1,3} & r_{3,3} & r_{3,4} \\ 0 & 0 & 2 & 2r_{3,3} \end{bmatrix} \end{aligned}$$

leaving the 18 entries

$$\begin{array}{cccccc} p_{1,3}, & p_{1,4}, & q_{1,3}, & q_{1,4}, & r_{1,3}, & r_{1,4}, \\ p_{2,3}, & p_{2,4}, & q_{2,3}, & q_{2,4}, & r_{2,3}, & r_{2,4}, \\ p_{3,3}, & p_{3,4}, & q_{3,3}, & q_{3,4}, & r_{3,3}, & r_{3,4} \end{array} \quad (13)$$

yet unknown. Now the first equation of (10) says that

$$\frac{1}{2}(8p_{2,4} - 5 + 4p_{3,3})x^2 + 4(2p_{1,4} - q_{3,3} - 2q_{2,4})xy - 4q_{1,4}y^2 + \cdots + (7q_{1,3} + \cdots + 2q_{2,3}r_{1,3})z^3$$

vanishes. Immediately, the coefficient of y^2 forces $q_{1,4} = 0$. More specifically, if

$$f(x, y, z) = 2xy + z^2 + x^2y - 2xz^2 \quad (14)$$

can be completed to a power series $F(x, y, z)$ defining a homogeneous hypersurface with isotropy, then any normalised Q will have $q_{1,4} = 0$. This will eventually be a consequence of the higher order terms and the normalisation (12). The coefficients of (10) give 41 polynomial constraints on (13). There are just two solutions, namely:

$$P = \begin{bmatrix} 5/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3/4 \\ 0 & 0 & 11/4 & 0 \\ 0 & 2 & 0 & 7/2 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 5/2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 5/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/4 & 0 \\ 0 & 2 & 0 & -3/2 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

These give rise to cases **I1.1** and **I1.2** of Theorem 2. Specifically, if we add a general quartic term

$$f(x, y, z) = 2xy + z^2 + x^2y - 2xz^2 + \sum_{i+j+k=4} c_{i,j,k} x^i y^j z^k$$

and now re-consider (9) with $N = 4$ and P, Q, R one of these two solutions of (10), then the quartic terms are determined. In fact, it is clear by inspection that (9) with $N = 4$ determines the quartic terms which only enter the right hand sides. The crucial observation, however, is that this overdetermined system is consistent as a consequence of (10). More precisely, the interpretation of (10) as the closure of a subalgebra of the Lie algebra of affine motions implies that the entire power series $F(x, y, z)$ may be defined implicitly by

$$\left(\exp \begin{bmatrix} rP + sQ + tR & r \\ & s \\ & t \\ & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ F(x, y, z) \\ 1 \end{bmatrix}. \quad (15)$$

Complete details for the analogous case of surfaces are in [2, §2].

To summarise then, cases **I1.1** and **I1.2** of Theorem 2 are the only possible completions of (14) defining a homogeneous hypersurface with isotropy (necessarily generated by (11)). It only remains to check that these hypersurfaces really do have this isotropy. For this, it suffices to take their 4th order truncations and apply Theorem 4 with $N = 4$ (without imposing the normalisations (12)). It turns out that P, Q, R are now determined by (9) alone up to adding arbitrary multiples of (11). Furthermore, (10) now holds. So this constitutes the full symmetry algebra and, apart from finding the explicit defining functions given in Theorem 2, cases **I1.1** and **I1.2** are complete. Notice that, because Theorem 4 applies directly when $N = 4$, the higher order terms are uniquely determined

simply by requiring the hypersurface to be homogeneous irrespective of whether it has isotropy. Finding explicit defining functions will be delayed until §4.

Though we can analyse all other cases in exactly the same way, there are some initial observations which almost immediately deal with some of them. Take, for example, the case **I2**. According to Theorem 3, the only possibility for isotropy in this case is the scaling (7) with $t = 2$. This limits the quartic terms to bx^4 for some b whilst all higher order terms must vanish. It is now easy to check that this equation does indeed define a homogeneous hypersurface with this isotropy. Moreover, since b is unaffected by the only residual coördinate freedom (namely, the isotropy), it is a true parameter.

In case **I3**, it may be that (7) with $t = 3$ survives in the isotropy of a corresponding hypersurface Σ . Straightaway this eliminates all terms higher than cubic and we have case **I3** of Theorem 2. However, there remains the possibility that (7) does not survive in the isotropy of Σ . This makes the isotropy one-dimensional, generated by the third matrix of (6) with $t = 0$. The corresponding one-parameter subgroup consists of null rotations (8) and in Theorem 2 we denote this case by **Inr**.

The detailed completion of cases **Inr** and **I0** follows the treatment of **I1** as above. The only real difficulty is in analysing the criteria (10) of Theorem 4. To ensure that all solutions of this system are found we employed Buchberger's algorithm for Gröbner bases as implemented in the 'grobner' package of MAPLE (Version V Release 3). In searching for homogeneous hypersurfaces with scaling isotropy (7) we can use (12) but in case **Inr** we use $p_{2,3} = q_{2,3} = r_{2,3} = 0$ instead. The entire analysis, including the calculation of 4th order and (if necessary) 5th order terms, can be completely automated. A MAPLE program is available by anonymous ftp[†]. Unlike **I1**, in most other cases the closure equations (10) have infinitely many solutions with some entries in P, Q, R remaining free. These free entries show up in the higher order terms of the corresponding completions as potential parameters. Having used the program **thm2proof** to find possible completions, there are two remaining tasks:

- apply the remaining coördinate freedoms from Theorem 3 to see whether they can be used to eliminate some of the parameters appearing in these possible completions;
- verify, by reapplying Theorem 4 with $N = 4$ or 5 , that these completions really do have the anticipated isotropy.

There are two cases when the first of these tasks is non-trivial. When there are no cubic terms, the typical output from **thm2proof** is

$$2xy + z^2 - 2p_{1,4}x^2y^2 - 2p_{1,4}xyz^2 - \frac{1}{2}p_{1,4}z^4.$$

[†]ftp://ftp.maths.adelaide.edu.au/pure/meastwood/maple/thm2proof

(The output can vary depending on the particular invocation of MAPLE because the ordering it uses for computing Gröbner bases etcetera depends on the internal addresses of the variables involved. This randomising effect can be used to advantage by running the program several times and choosing the simplest answer.) The rescaling (3) corresponding to the first of (6), has the effect of multiplying the coefficient of z^4 by λ^2 . If this coefficient is non-zero, we may therefore normalise it to unity and obtain case **Sp**. On the other hand, if it is zero then we obtain case **Qd**. The other cases requiring special attention are **I3** and **Inr** when x^3 is the only cubic term. No matter what isotropy is assumed, the only possible quartic term is a multiple of x^4 . A non-zero multiple may be normalised to x^4 itself by a suitable rescaling (7) with $t = 3$. This cuts down the residual coördinate freedom to (8) and leads to case **Inr**. When there are no quartic terms, then we are led to **I3**. The final task of verifying that these hypersurfaces really do have isotropy and, indeed, computing the full symmetry algebra and checking that it closes is accomplished with a separate MAPLE program[‡].

4 Proof of Theorem 1

There are two possible ways to proceed. We could start with the list of explicit defining functions, verify that each of them gives a homogeneous surface with isotropy, and then execute the normalisations of §2 to obtain a perfect match with Theorem 2. We have written a MAPLE program[§] which takes a defining function, computes its prospective symmetry algebra (by truncating its power series as in Theorem 4), checks that this algebra closes, and then determines whether the hypersurface is genuinely invariant under these symmetries (to infinite order). The 20 possibilities of the list are already in **thm1** and, indeed, this program shows them to be homogeneous with isotropy. Of course, this approach is somewhat unsatisfactory because it does not explain where the list comes from nor why minor variations such as

$$\begin{array}{lll} W = XY + Z^2 \log Z & WZ = XY + e^Z & WZ = XY + \log Z \\ W^2 = XY + \log Z & W^2 = XY + Z \log Z & W^2 = XY + Z^2 \log Z \end{array}$$

are omitted. (According to Theorem 1, they would already be on the list but for a different choice of coördinates.) In fact, none of these is homogeneous as **thm1** readily verifies. For example, the first of them truncated at 4th order defines a closed symmetry algebra but does not satisfy this algebra at 5th order. Rather,

$$W + \frac{1}{28} + \frac{2}{7}Z = XY + \frac{9}{28}\left(\frac{5}{3}Z - \frac{2}{3}\right)^{12/5}$$

[‡][ftp://ftp.maths.adelaide.edu.au/pure/meastwood/maple/thm2verify](http://ftp.maths.adelaide.edu.au/pure/meastwood/maple/thm2verify)

[§][ftp://ftp.maths.adelaide.edu.au/pure/meastwood/maple/thm1](http://ftp.maths.adelaide.edu.au/pure/meastwood/maple/thm1)

is a homogeneous surface (**#7** with an affine change of coördinates) which just happens to have the same power series expansion about the point $(0, 0, 0, 1)$ up to 4th order.

More satisfactory is to start with Theorem 2 and derive explicit defining functions in each case. The matrices P, Q, R supplied by **thm2verify** describe the hypersurfaces parametrically (15) and, with sufficient diligence, it is possible explicitly to solve for $F(x, y, z)$ and, after a suitable change of coördinates, check the comparison table given in §1. There are, however, some observations which greatly simplify this task. Cases **Qd** and **Sp** are clear by inspection (since they are manifestly homogeneous and have the correct power series expansion up to 4th order). Cases **I3** and **I2** are also immediate: it was already observed in §3 that their defining functions must be polynomial.

In cases **I1** having (11) generating the isotropy forces the defining function $F(x, y, z)$ to have the form

$$F(x, y, z) = f(x)y + g(x)z^2.$$

The output from **thm2verify** has

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} + \text{isotropy}$$

in both **I1** cases. That the corresponding vector field

$$\left(1 - \frac{x}{2}\right) \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial w}$$

be a symmetry implies that $f(x) = 4x/(2 - x)$. The vector fields corresponding to R distinguish **I1.1** from **I1.2**:

$$2z \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2z \frac{\partial}{\partial w} \quad \text{versus} \quad -\frac{z}{2} \frac{\partial}{\partial y} + \left(1 + \frac{5}{2}x\right) \frac{\partial}{\partial z} + 2z \frac{\partial}{\partial w}.$$

They determine $g(x)$ as $(2 - 5x)/(2 - x)$ or $4/((2 - x)(2 + 5x))$, respectively. These are the defining functions given in Theorem 2 and the affine coördinate changes

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 & -2/5 & 0 & 0 \\ -1 & 0 & -5 & 0 \\ 0 & 0 & 0 & 2 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} W \\ X \\ Y \\ Z \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 & 2/5 & 0 & 0 \\ 2 & -2 & -7 & -6 \\ 2 & -2 & 0 & 0 \\ -8 & 8 & 8 & 4 \end{bmatrix} \begin{bmatrix} W \\ X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} -2/5 \\ 6 \\ 0 \\ -4 \end{bmatrix}$$

give the defining functions **#5** and **#6** with their respective basepoints.

Case **I_{nr}** has

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} + \text{isotropy}$$

which implies that

$$F(x, y, z) = z^2 + \theta(x, y)$$

and it follows easily that $\{w = \theta(x, y)\}$ is a homogeneous surface in affine three-space. These were classified in [1, 2, 4] and in [2] a method was given for locating any given surface. In fact, it is easy to spot that these surfaces are exactly the class **N6** of [2, Theorem 2] with almost the same normalisation: the parameter b is exactly as in [2]. Furthermore, in [2, §6.2] was given a precise comparison between these normal forms and the explicit defining functions of [1]. Following this through gives ~~#16–#20~~ as in the comparison table of §1. (Though in [1] the link between symmetry algebra #9 and surface #12 should have α replaced by $1/\alpha$.)

There is a similar link with homogeneous surfaces in cases **I0**. The isotropy implies that

$$F(x, y, z) = \theta(u, z) \quad \text{where } u = xy$$

whence

$$\begin{aligned} \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, -1 \right] & \left[\begin{bmatrix} p & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ F \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ r \\ s \end{bmatrix} \right] \\ &= \left[\frac{\partial \theta}{\partial u}, \frac{\partial \theta}{\partial z}, -1 \right] \left[\begin{bmatrix} p+q & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \begin{bmatrix} u \\ z \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ r \\ s \end{bmatrix} \right] \\ \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, -1 \right] & \left[\begin{bmatrix} 0 & 0 & r & s \\ 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & q & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ F \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} \right] \\ &= y \left[\frac{\partial \theta}{\partial u}, \frac{\partial \theta}{\partial z}, -1 \right] \left[\begin{bmatrix} 0 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ z \\ \theta \end{bmatrix} + \begin{bmatrix} t \\ p \\ q \end{bmatrix} \right] \\ \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, -1 \right] & \left[\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r & s \\ p & 0 & 0 & 0 \\ q & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ F \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} \right] \\ &= x \left[\frac{\partial \theta}{\partial u}, \frac{\partial \theta}{\partial z}, -1 \right] \left[\begin{bmatrix} 0 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ z \\ \theta \end{bmatrix} + \begin{bmatrix} t \\ p \\ q \end{bmatrix} \right]. \end{aligned}$$

It turns out that the matrices P, Q, R in cases **I0** have the forms indicated in

these identities. For example, in case **10.1** with $b = 6$ **thm2verify** gives

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3/2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3/2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 15/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -9/8 \\ 0 & 0 & 2 & 9 \end{bmatrix}$$

apart from isotropy. Therefore, the surface $\{w = \theta(u, z)\}$ is homogeneous and, following the notation of [1], its symmetry algebra contains

$$\begin{bmatrix} 15/2 & 0 & 0 \\ 0 & 6 & -9/8 \\ 0 & 2 & 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -3/2 \\ 2 \end{bmatrix}. \quad (16)$$

The second of these generates a uniform translation so the surface is a cylinder, i.e. class **D2** of [2]. From the full symmetry algebra of **D2** given in [2] it is easy to check that the true parameter a^2 , if non-zero, is given by

$$\frac{32[3 \operatorname{tr}(M) \operatorname{tr}(M^2) - 2 \operatorname{tr}(M^3) - 3 \operatorname{tr}(M^2)\lambda - 3 \operatorname{tr}(M)\lambda^2 + 5\lambda^3]}{25[\operatorname{tr}(M) - \lambda][5 \operatorname{tr}(M)^2 - 9 \operatorname{tr}(M^2) - 10 \operatorname{tr}(M)\lambda + 14\lambda^2]}$$

for any non-zero M from the matrix part of the algebra where λ is the eigenvalue of M for the translation vector. Thus $a^2 = 64/25$ and from [2, §1] the surface must be $\{Z = X \log X\}$ for a suitable choice of affine coördinates.

Unfortunately, this abstract reasoning loses track of the distinguished coördinate $u = xy$ so a more direct argument must be employed. The change of coördinates

$$\begin{bmatrix} u \\ z \\ w \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/2 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} u \\ z \\ w \end{bmatrix}$$

preserves u but conjugates the symmetries (16) to

$$\begin{bmatrix} 15/2 & 0 & 0 \\ 0 & 15/2 & 1 \\ 0 & 0 & 15/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 3/4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Now we can employ the surface version of (15):

$$\left(\exp \begin{bmatrix} 15s/2 & 0 & 0 & t \\ 0 & 15s/2 & s & s/2 + t \\ 0 & 0 & 15s/2 & 3s/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10}se^{15s/2} + \frac{2}{75}(2s+5t)(e^{15s/2}-1)/s \\ \frac{2}{15}t(e^{15s/2}-1)/s \\ \frac{1}{10}(e^{15s/2}-1) \\ 1 \end{bmatrix} = \begin{bmatrix} u \\ z \\ w \\ 1 \end{bmatrix}.$$

This may be solved:

$$75z = 75u + (1 + 10w) \log(1 + 10w) + 40w$$

and a further affine change of coördinates

$$W = 75z - 40w \quad X = 75x \quad Y = y \quad Z = 1 + 10w$$

evidently gives **#10**. All other cases follow similarly.

5 Remarks on the real case

The analysis in the real case proceeds as for the complex case save for a few minor changes. Notice that in §3, only twice was it used that we were working over the complex numbers. It was when we were normalising the quartic terms in cases **Sp** and **Inr**. Taking this into account, it follows that these cases have two real forms:

$$\begin{aligned} \mathbf{Sp}^\pm \quad w &= 2xy + z^2 \pm 4x^2y^2 \pm 4xyz^2 \pm z^4 + O(5) = \pm(1 - \sqrt{1 \mp 4(2xy + z^2)})/2 \\ \mathbf{Inr}^\pm \quad w &= 2xy + z^2 + x^3 \pm x^4 + bx^5 + O(6) \end{aligned}$$

Therefore, the conclusion to be drawn from §3 in the real case is that, with these two exceptions, if we start off a real power series as $w = 2xy + z^2 + \dots$ and continue with no cubic terms or with cubic terms listed in Theorem 3, then the resulting list of real homogeneous hypersurfaces with isotropy is just as in Theorem 2.

However, before arriving at §3 we were normalising the defining equations in §2 and here also, complex numbers entered at two stages. The first was in normalising the quadratic terms. Over the reals there are two possibilities, namely

$$w = 2xy + z^2 + O(3) \quad \text{and} \quad w = x^2 + y^2 + z^2 + O(3),$$

hyperbolic and *elliptic*. This now shows up in the second stage where we conjugated and rescaled a matrix in $\mathfrak{o}(3, \mathbb{C})$ into one of the three standard forms (5). For $\mathfrak{o}(2, 1)$ there are four standard forms because non-null vectors now come in two flavours, either space-like or time-like. Over $O(3, \mathbb{C})$ the corresponding matrices are conjugate up to scale:

$$\sqrt{2}i \begin{bmatrix} i/\sqrt{2} & i/\sqrt{2} & 1 \\ i/\sqrt{2} & i/\sqrt{2} & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i/\sqrt{2} & i/\sqrt{2} & 1 \\ i/\sqrt{2} & i/\sqrt{2} & -1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

but not over $O(2, 1)$. As a corresponding complex coördinate change we may choose

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} i/\sqrt{2} & i/\sqrt{2} & 1 \\ i/\sqrt{2} & i/\sqrt{2} & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad w \mapsto -2w$$

which takes $w = 2xy + z^2 + 3xyz - z^3 + \dots$ to

$$w = 2xy + z^2 + \frac{5}{4}x^3 - \frac{3}{4}(x^2y - 2xz^2) + \frac{3}{4}(xy^2 - 2yz^2) - \frac{5}{4}y^3 + \dots$$

and gives two real forms for each of **I0**. This is the full extent of this alternative normalisation. So cases **Sp**, **Inr**, and **I0** in Theorem 2 have two hyperbolic real forms and the rest have just one.

It is easy to check that the Pick invariant is zero in cases **I3**, **I2**, and **I1** of Theorem 3. Therefore, none of the corresponding hypersurfaces in Theorem 2

(including case **Inr**) can have an elliptic real form. The complex change of coördinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

gives the unique elliptic form of case **Qd** and cases **I0**. It gives one of the two real forms of **Sp**, namely the sphere. The other is the hyperhyperboloid of two sheets. Assembling these observations and tracing through to the explicit defining functions gives the following real classification list.

#1	$W = XY + Z^2$	$W = X^2 + Y^2 + Z^2$
#2	$W^2 = XY \pm Z^2 + 1$	$\pm W^2 = X^2 + Y^2 + Z^2 \pm 1$
#3	$W = XY + Z^2 + X^3$	
#4	$W = XY + Z^2 + X^2Z + \alpha X^4$	
#5	$W = XY + Z^2 + XZ^2$	
#6	$W^2 = XY + X^2Y + X^2Z$	
#7	$W = XY + Z^\alpha$	$W = X^2 + Y^2 \pm Z^\alpha$
#8	$W = XY + e^Z$	$W = X^2 + Y^2 \pm e^Z$
#9	$W = XY + \log Z$	$W = X^2 + Y^2 \pm \log Z$
#10	$W = XY + Z \log Z$	$W = X^2 + Y^2 \pm Z \log Z$
#11	$W^2 = XY + Z^\alpha$	$W^2 = X^2 + Y^2 \pm Z^\alpha$
#12	$W^2 = XY + e^Z$	$W^2 = X^2 + Y^2 \pm e^Z$
#13	$WZ = XY + Z^\alpha$	$WZ = X^2 + Y^2 \pm Z^\alpha$
#14	$WZ = XY + Z \log Z$	$WZ = X^2 + Y^2 \pm Z \log Z$
#15	$WZ = XY + Z^2 \log Z$	$WZ = X^2 + Y^2 \pm Z^2 \log Z$
#16	$W = XY \pm Z^2 + X^\alpha$	
#17	$W = XY \pm Z^2 + e^X$	
#18	$W = XY \pm Z^2 + \log X$	
#19	$W = XY \pm Z^2 + X \log X$	
#20	$W = XY \pm Z^2 + X^2 \log X$	

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